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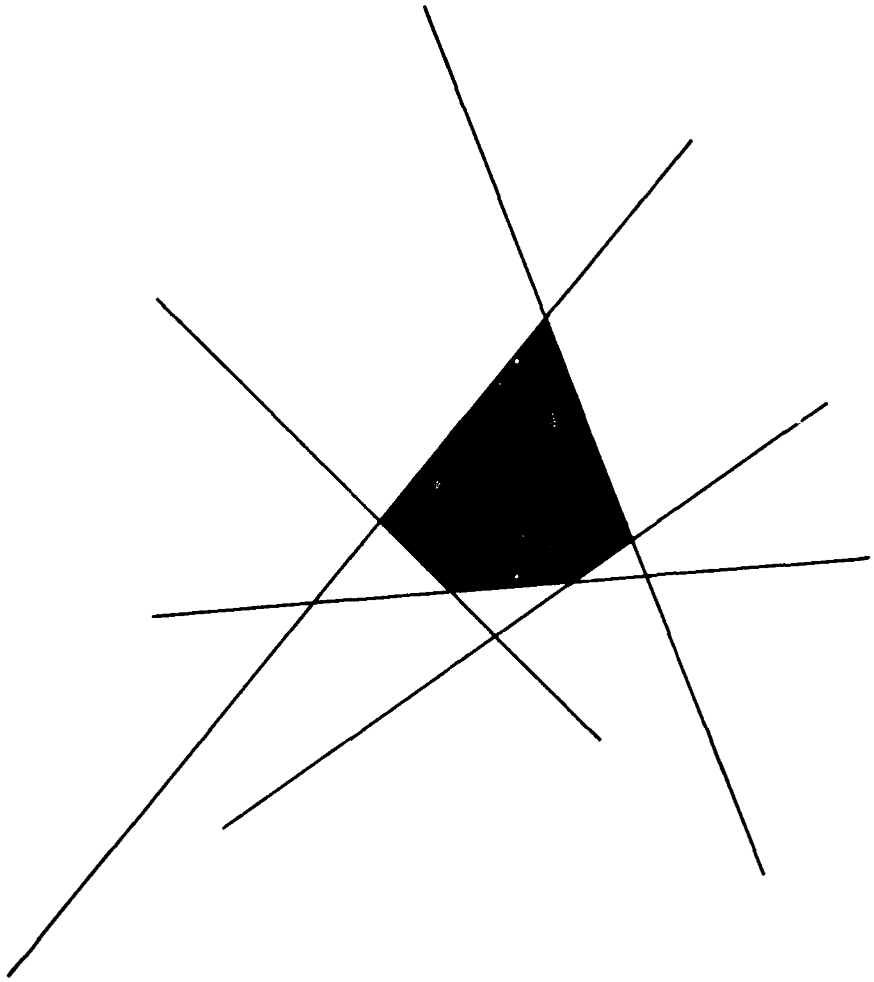
by

NOZER D. SINGPURWALLA



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AN ADAPTIVE BAYESIAN SCHEME FOR ESTIMATING RELIABILITY

GROWTH UNDER EXPONENTIAL FAILURE TIMES

by

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DECEMBER 1979 ORC 79-17

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ABSTRACT

In this paper we consider an adaptive approach for estimating reliability growth, based on prior information which is motivated from practical considerations. We discuss two situations: in the first one, both the prior distribution and the posterior distributions of the mean time to failure of an exponential distribution are stochastically ordered; in the second situation, the prior distribution is stochastically ordered with respect to the last posterior distribution. The former situation leads us to a procedure which is not fully Bayesian, and is therefore termed by us as "pseudo-Bayesian." Since we do not know the properties of this pseudo Bayesian approach, we can best describe our work here as being a "pseudo-Bayesian scheme." The second situation leads us to an approach which is fully Bayesian under certain assumptions. Our work in this general area of reliability growth is still in progress, and we invite the attention of other researchers to look into some of the problems that we have posed, and the questions that we have raised.

AN ADAPTIVE BAYESIAN SCHEME FOR ESTIMATING RELIABILITY GROWTH UNDER EXPONENTIAL FAILURE TIMES

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Nozer D. Singpurwalla

1. INTRODUCTION

A complex, newly designed system, generally undergoes several stages of testing before it is put into operation. After each stage of testing, changes are made to the design (or the operating conditions are re-specified) with the hope that the new design would lead to a longer period of performance. This procedure is referred to as "reliability growth"; this is because a longer period of performance implies an improvement in reliability. Often, in practice, a change in the system design may result in a deterioration of the system performance, so that the term reliability growth may not be an appropriate description of what is actually happening to the system. However, since the intent of a design change is to improve the reliability, we shall continue to use the term reliability growth throughout this paper.

Suppose then, that the system has been tested at stages $0,1,2, \ldots, \tau$. At the end of each stage, an estimate of the reliability of the system is made using any of the conventional procedures which measure the reliability of the system. At each stage of testing, we may test either a single copy of the system, or several copies of the system, depending on the particular situation being considered. Having tested the system over $(\tau + 1)$ stages, we would like to know the following:

i) Has there been a genuine growth in reliability over the time period covered by the $(\tau + 1)$ stages of testing, and is this growth significant?

ii) Using all the testing information that we have acquired over all of the stages, what is our best (in a sense to be made precise later on) estimate of the reliability of the system at stage τ?

An answer to question i) will enable a decision maker to determine if indeed the design changes do result in an improvement in reliability, and perhaps determine the rate (with respect to the stages) at which there is a growth in reliability. An answer to question ii) will enable a decision maker to decide if the system is ready to be put into operation, and to arrive at a suitable cost-warranty agreement with the user. We can, of course envision several other uses of the analysis described in i) and ii) above.

Since 1966, there have been several papers written under the heading of reliability growth, each emphasizing a different point of view. For a recent survey of these papers, we refer the reader to H. Balaban (1978). The approach that we shall take in this paper is Bayesian, with our Bayesian analysis being undertaken in a manner which will be described in Section 2. Others who have adopted a Bayesian point of view, but with a direction different from ours are A.F.M. Smith (1977) and Weinrich and A. Gross (1978).

2. A BAYESIAN SCHEME FOR ESTIMATING RELIABILITY GROWTH

We shall start off by considering the following model:

Suppose that the failure distribution of the system after the i^{th} design change, $i = 0,1, \ldots, \tau$, is an exponential distribution with a mean θ_i . That is

$$f(t;\theta_i) = \frac{1}{\theta_i} e^{-t/\theta_i} , \theta_i > 0 , t > 0 .$$

In particular, at stage 0, when the system is newly built, its time to failure is an exponential distribution with mean θ_0 . Based upon our previous knowledge of the mean time to failure of similarly designed systems, we shall assign a prior distribution to θ_0 , say $G(\theta_0;\cdot)$. Without any loss in generality, we shall let $G(\theta_0;\cdot)$ be the natural conjugate prior, which in this case is an *inverted gamma* distribution, with parameters α_0 and β_0 that is

$$(2.0) \quad dG(\theta_0;\cdot) = g(\theta_0;\alpha_0,\beta_0) = \frac{\alpha_0^{\beta_0}}{\Gamma(\beta_0)} \left(\frac{1}{\theta_0}\right)^{\beta_0+1} e^{-\alpha_0/\theta_0} , \text{ for } \alpha_0 , \beta_0 , \theta_0 > 0 .$$

It is easy to verify that the prior mean is

$$E(\theta_0) = \frac{\alpha_0}{\beta_0 - 1} < \infty , \text{ if } \beta_0 > 1 ;$$

thus we shall require that $\beta_0 > 1$.

Having assigned our prior distribution $G(\theta_0;\cdot)$, we shall test n_0 (≥ 1) copies of the system, and observe their times to failure,

$t_{0,i}$, $i = 1,2, \ldots, n_0$. Let $T_0 \stackrel{\text{def}}{=} \sum_{i=1}^{n_0} t_{0,i}$, be the observed *total time on test.*

Conditioned on T_0, the posterior distribution of θ_0 under the inverted gamma prior is also an inverted gamma. That is

$$dG(\theta_0 \mid T_0;\cdot) = g(\theta_0 \mid T_0;\alpha_0,\beta_0,n_0)$$

(2.1)
$$= \frac{(\alpha_0 + T_0)^{\beta_0+n_0}}{\Gamma(\beta_0 + n_0)} \left(\frac{1}{\theta_0}\right)^{\beta_0+n_0+1} e^{-(\alpha_0+T_0)/\theta_0} .$$

The mean of the posterior distribution of θ_0, is

(2.2)
$$E(\theta_0 \mid T_0) = \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} .$$

The above quantity is our Bayes estimate of the mean time to failure at stage 0, based on the prior distribution $g(\theta_0;\alpha_0,\beta_0)$, the total time on test T_0, and assuming a squared error loss function. A posterior $(1 - \gamma)\%$ credibility interval for θ_0 can be obtained by finding two numbers $\underline{(\theta_0 \mid T_0)}$ and $\overline{(\theta_0 \mid T_0)}$ such that

(2.3)
$$\int_{\underline{(\theta_0|T_0)}}^{\overline{(\theta_0|T_0)}} g(\theta_0 \mid T_0;\alpha_0,\beta_0,n_0)d\theta_0 = (1 - \gamma) .$$

The parameter θ_0 and its posterior distribution is a natural measure of the performance of the system at stage 0. Having observed the mean of the posterior distribution of θ_0, we may want to increase it by making either design changes or re-specifying the operating conditions of the system.

Suppose that we do decide to make these changes; let θ_1 denote the mean time to failure after the changes have been made. The system is now at stage 1, and we are ready to test the system and verify if the changes have resulted in an improvement of the system.

Since we have adopted a Bayesian point of view, our first task is to choose a prior distribution for θ_1, say $G(\theta_1;\cdot)$. The novelty of our approach pertains to the manner in which we go about choosing $G(\theta_1;\cdot)$, and this is motivated by the following consideration:

Even though the changes that we have instituted have been undertaken with a view towards increasing the reliability of the system, there is a possibility that the changes could be deleterious to the system. In order to account for this possibility we shall choose $G(\theta_1;\cdot)$ in such a manner that θ_1 is *stochastically larger* than $(\theta_0 \mid T_0)$; this is written as $\theta_1 \overset{st}{\geq} (\theta_0 \mid T_0)$. Thus, we must have

$$P(\theta_1 \geq x) \geq P((\theta_0 \mid T_0) \geq x) \text{ , for all } x \geq 0 \text{ .}$$

The usual strategy for situations of this type is to take $\theta_1 \geq (\theta_0 \mid T_0)$ with probability 1, as has been done by Barlow et al. (1972) and by Smith (1977). However, a prior chosen to satisfy the above condition does not place any mass in the region $\theta_1 < (\theta_0 \mid T_0)$, and therefore would exclude from our analysis the possibility of an adverse effect of the changes. Hence, for the situation that we are considering here, the requirement that $\theta_1 \overset{st}{\geq} (\theta_0 \mid T_0)$ makes more practical sense.

Following our previous discussion, suppose that the prior distribution $G(\theta_1;\cdot)$ is also an inverted gamma with parameters α_1 and β_1. That is,

$$(2.4)\quad dG(\theta_1;\cdot) = g(\theta_1;\alpha_1,\beta_1) = \frac{\alpha_1^{\beta_1}}{\Gamma(\beta_1)}\left(\frac{1}{\theta_1}\right)^{\beta_1+1} e^{-\alpha_1/\theta_1}\ , \text{ for } \alpha_1\ ,\ \beta_1\ ,\ \theta_1 > 0\ .$$

A sufficient condition for ensuring that $\theta_1 \overset{st}{\geq} (\theta_0 \mid T_0)$ is to have $\alpha_1 \geq \alpha_0 + T_0$, and $\beta_1 = \beta_0 + n_0$. Verify that under the above conditions $E(\theta_1) \geq E(\theta_0 \mid T_0)$, a condition that we would hope to achieve under the hypothesis that a change is beneficial to the system. A natural strategy is to choose $\alpha_1 = \alpha_0 + T_0 + a_1$, where the value of a_1 reflects our prior belief about the magnitude of the improvement in the mean as a result of the changes. When α_1 is thus chosen, it is *important* that at stage 1 we treat T_0 as being a *constant*. If this is not done, then the posterior distribution of θ_1 will have to be obtained by treating α_1 as a hyper-parameter with a prior distribution which is related to the unconditional distribution of T_0 . Under the above circumstances, the posterior distribution of θ_1 will not be an inverted gamma, and our procedure will become computationally involved. Furthermore, we shall also assume that $g(\theta_1;\alpha_1,\beta_1)$ is *independent* of $g(\theta_0 \mid T_0;\cdot)$.

Having chosen $g(\theta_1;\alpha_1,\beta_1)$, we test $n_1 \geq 1$ copies of the system (under stage 1), and observe T_1 , the corresponding total time on test. Since we have taken T_0 to be a constant (once we are at stage 1) the posterior distribution of θ_1 conditioned on T_1 is

$$(2.5)\quad g(\theta_1 \mid T_1;\cdot) = \frac{(\alpha_0 + T_0 + a_1 + T_1)^{\beta_0+n_0+n_1}}{\Gamma(\beta_0 + n_0 + n_1)}\left(\frac{1}{\theta_1}\right)^{\beta_0+n_0+n_1-1} e^{-(\alpha_0+T_0+a_1+T_1)}$$

with the parameters having the usual positive signs.

Because of independence, and under the assumption of a squared error loss, the Bayes estimator of θ_1 is the mean of the posterior distribution of θ_1

$$E(\theta_1 \mid T_1) = \frac{\alpha_0 + T_0 + a_1 + T_1}{\beta_0 + n_0 + n_1 - 1} . \tag{2.6}$$

Under our postulate of reliability growth, we would want to have

$$(\theta_1 \mid T_1) \overset{st}{\geq} (\theta_0 \mid T_0) ;$$

a *necessary* (though not sufficient) condition for the above inequality is that

$$\frac{\alpha_0 + T_0 + a_1 + T_1}{\beta_0 + n_0 + n_1 - 1} \geq \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} , \tag{2.6.1}$$

which reduces to the requirement that

$$\frac{a_1 + T_1}{n_1} \geq \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} . \tag{2.7}$$

If Equation (2.7) is *not satisfied*, we proceed to Section 2.1 and follow the strategies discussed there. If Equation (2.7) is *satisfied*, then we can either use $E(\theta_1 \mid T_1)$ as our Bayes estimator of θ_1 , with a $(1 - \gamma)\%$ credibility interval for θ_1 given by the two numbers $\underline{(\theta_1 \mid T_1)}$ and $(\overline{\theta_1 \mid T_1})$ such that

$$\int_{\underline{(\theta_1|T_1)}}^{(\overline{\theta_1|T_1})} g(\theta_1 \mid T_1;\cdot)d\theta_1 = 1 - \gamma ,$$

and stop, or we can proceed to stage 2 by making the appropriate modifications to the system.

If we proceed to stage 2, then the prior distribution of θ_2 should be such that

$$\theta_2 \overset{st}{\geq} (\theta_1 \mid T_1) \; . \tag{2.8}$$

One way of achieving Condition (2.8) is to choose α_2 and β_2, the parameters of the inverted gamma prior distribution of θ_2 in such a manner that $\alpha_2 = \alpha_0 + T_0 + a_1 + T_1 + a_2$ and $\beta_2 = \beta_0 + n_0 + n_1$. As before, a_2 reflects our belief about the magnitude of the improvement in the mean as a result of the changes, and T_0 and T_1 are now assumed to be constants. We shall continue with our discussion of this strategy in Section 2.2.

2.1 Procedures When the Posterior Distributions at Stages 0 and 1 Are Not Stochastically Ordered

If Equation (2.7) is not satisfied, that is, if

$$\frac{a_1 + T_1}{n_1} < \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} \; , \tag{2.9}$$

and if we have *no* reason to believe that the reliability growth postulate may *not* be true, then we conclude that the inequality (2.9) is brought about by the randomness in observing T_0 and T_1. In order to rectify this, we propose two strategies, each having its own peculiarities:

Strategy 1:

Ignore the inequality (2.9), and *for the time being* accept the result that $E(\theta_1 \mid T_1) < E(\theta_0 \mid T_0)$; proceed to stage 2, by choosing the prior

distribution of θ_2 in such a manner that Equation (2.8) is satisfied. After completing the testing over all the $\tau + 1$ stages, we will perform an isotonic regression of the posterior means; this will be discussed in Section 3.

Strategy 2:

By ignoring the inequality (2.9) and directly proceeding to stage 2, we will, through the prior distribution of θ_2, allow the effects of (2.9) to perpetuate over the succeeding stages. We can *avoid* this by pooling the violators T_0 and T_1 and also the n_0 and n_1. That is,

- we replace both T_0 and T_1 by $T_{01} \overset{\text{def}}{=} \frac{T_0 + T_1}{2}$, and
- we replace both n_0 and n_1 by $n_{01} \overset{\text{def}}{=} \frac{n_0 + n_1}{2}$.

If we pool as above, then, we must test to see if

$$\frac{a_1 + T_{01}}{n_{01}} \geq \frac{\alpha_0 + T_{01}}{\beta_0 + n_{01} - 1} \tag{2.10}$$

which is a condition analogous to (2.7) except that the n_i and the T_i, $i = 0,1$, have been replaced by their pooled values. Since $(\beta_0 - 1) > 0$, a condition which is needed to ensure that the mean of the prior distribution of θ_0 is finite, a sufficient condition for (2.10) is that

$$a_1 \geq \alpha_0 .$$

To summarize, we must choose $a_1 \geq \alpha_0$, and if the inequality (2.7) is violated, then we shall pool and be ensured that

$$(2.11) \qquad \frac{\alpha_0 + T_{01} + a_1 + T_{01}}{\beta_0 + n_{01} + n_{01} - 1} \geq \frac{\alpha_0 + T_{01}}{\beta_0 + n_{01} - 1} ,$$

in lieu of our original requirement that

$$\frac{\alpha_0 + T_0 + a_1 + T_1}{\beta_0 + n_0 + n_1 - 1} \geq \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} .$$

Since Equation (2.11) can also be written as

$$(2.12) \qquad \frac{\alpha_0 + T_0 + T_1 + a_1}{\beta_0 + n_0 + n_1 - 1} \geq \frac{2\alpha_0 + T_0 + T_1}{2\beta_0 + n_0 + n_1 - 1} ,$$

and since the left hand side of the above equation is identical to the left hand side of Equation (2.6.1), the effect of pooling is to lower the magnitude of the right hand side of (2.6.1). Recall that the right hand side of (2.6.1) is the posterior mean of θ_0 given T_0 . *Thus, the effect of pooling is to lower the posterior mean at the previous stage in the light of the information obtained at the current stage, the previous stage, and the prior assumption.*

Thus, our revised Bayes estimator of θ_0 is

$$(2.13) \qquad E(\theta_0 \mid T_0, T_1) = \frac{2\alpha_0 + T_0 + T_1}{2\beta_0 + n_0 + n_1 - 1} ,$$

and the revised credibility intervals for θ_0 are given by

$$\int_{\underline{(\theta_0|T_0,T_1)}}^{\overline{(\theta_0|T_0,T_1)}} g(\theta_0 \mid T_0, T_1; \cdot) d\theta_0 = 1 - \gamma$$

where $g(\theta_0 \mid T_0,T_1;\cdot)$ is an inverted gamma distribution with a scale parameter $(2\alpha_0 + T_0 + T_1)/2$ and a shape parameter $(2\beta_0 + n_0 + n_1)/2$.

Our Bayes estimator of θ_1 remains unchanged and is given by

$$(2.14) \qquad E(\theta_1 \mid T_1) = \frac{\alpha_0 + T_0 + T_1 + a_1}{\beta_0 + n_0 + n_1 - 1}$$

with credibility intervals given by the procedure which follows Equation (2.7).

We can stop at this point, or proceed to stage 2 by choosing an inverted gamma prior distribution of θ_2 with prior parameters α_2 and β_2 where $\alpha_2 = \alpha_0 + T_0 + T_1 + a_1 + a_2$ and $\beta_2 = \beta_0 + n_0 + n_1$, with T_0 and T_1 treated as being constants.

2.2 Analysis of Failure Data at Stage 2

We have seen before, that irrespective of whether condition (2.7) is satisfied or not, the prior distribution of θ_2 will be an inverted gamma with parameters α_2 and β_2 , where

$$(2.15) \qquad \alpha_2 = \alpha_0 + T_0 + a_1 + T_1 + a_2 \quad \text{and} \quad \beta_2 = \beta_0 + n_0 + n_1 .$$

Having chosen $g(\theta_2;\alpha_2,\beta_2)$, we test $n_2 \geq 1$ copies of the system (under stage 2), and observe T_2 the corresponding total time on test. If $g(\theta_2;\alpha_2,\beta_2)$ is assumed to be independent of $g(\theta_1 \mid T_1;\cdot)$, then the posterior distribution of θ_2 given T_2 is also an inverted gamma with parameters $(\alpha_0 + T_0 + a_1 + T_1 + a_2 + T_2)$ and $(\beta_0 + n_0 + n_1 + n_2)$.

Here again, in order to be assured that

$$(\theta_2 \mid T_2) \overset{st}{\geq} (\theta_1 \mid T_1)$$

we shall need, as a *necessary condition*,

$$\frac{a_2 + T_2}{n_2} \geq \frac{\alpha_0 + T_0 + a_1 + T_1}{\beta_0 + n_0 + n_1 - 1} \quad (2.16)$$

If condition (2.16) is satisfied, then, we can either stop and because of the independence assumption use $(\alpha_0 + T_0 + a_1 + T_1 + a_2 + T_2)/(\beta_0 + n_0 + n_1 + n_2 - 1)$ as our Bayes estimator of θ_2, with a $100(1 - \gamma)\%$ credibility interval for θ_2 given by the two numbers $(\underline{\theta}_2, T_2)$ and $(\bar{\theta}_2, T_2)$, such that

$$\int_{(\underline{\theta}_2, T_2)}^{(\bar{\theta}_2, T_2)} g(\theta_2 \mid T_2; \cdot)\, d\theta_2 = 1 - \gamma ,$$

or we can proceed to stage 3 by making the appropriate modifications to the system. If we proceed to stage 3, we repeat our cycle so that the prior distribution of θ_3 satisfies the condition

$$\theta_3 \overset{st}{\geq} (\theta_2 \mid T_2) .$$

If condition (2.16) is *not satisfied*, then we can, following Strategy 1 of Section 2.1, ignore the inequality (2.16) and proceed directly to stage 3, or follow Strategy 2 and pool T_1 and T_2 to obtain $T_{12} \overset{def}{=} (T_1 + T_2)/2$, and $n_{12} \overset{def}{=} (n_1 + n_2)/2$. If we choose to pool, then, analogous to (2.7) and (2.10), we must have

$$\frac{a_2 + T_{12}}{n_{12}} \geq \frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} . \quad (2.17)$$

A *sufficient* condition for the above is that $a_2 \geq \alpha_0 + T_0 + a_1$. *Thus, at this stage, the prior parameter* a_2 *has to bear a relationship to* T_0 *, the total time on test at stage 0.*

As we emphasized in Section 2.1, the effect of pooling is a lowering of the posterior mean at the previous stage. In the present case, we have changed the posterior mean at stage 1 from $(\alpha_0 + T_0 + a_1 + T_1)/(\beta_0 + n_0 + n_1 - 1)$ (Equation (2.14)) to

$$(2.18) \quad E(\theta_1 \mid T_0,T_1,T_2) = \frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} = \frac{2(\alpha_0 + T_0 + a_1) + T_1 + T_2}{2(\beta_0 + n_0) + n_1 + n_2 - 1} .$$

Furthermore, our credibility intervals for θ_1 will now be given by

$$\int_{\underline{(\theta_1 | T_0,T_1,T_2)}}^{\overline{(\theta_1 | T_0,T_1,T_2)}} g(\theta_1 \mid T_0,T_1,T_2;\cdot)d\theta_1 = 1 - \gamma$$

where $g(\theta_1 \mid T_0,T_1,T_2)$ is an inverted gamma distribution with a scale parameter $(2(\alpha_0 + T_0 + a_1) + T_1 + T_2)/2$ and a shape parameter $(2(\beta_0 + n_0) + n_1 + n_2 - 1)/2$.

Of course, our Bayes estimator of θ_2 remains unchanged as $(\alpha_0 + T_0 + a_1 + T_1 + a_2 + T_2)/(\beta_0 + n_0 + n_1 + n_2 - 1)$.

Since we have revised our estimator of θ_1 from that given by Equation (2.14) to that given by Equation (2.18), we will have to see if

$$(2.19) \quad (\theta_1 \mid T_0,T_1,T_2) \overset{st}{\geq} (\theta_0 \mid T_0)$$, if we *did not* have to pool at stage 1,

or

$$(2.19.1) \quad (\theta_1 \mid T_0,T_1,T_2) \overset{st}{\geq} (\theta_0 \mid T_0,T_1)$$, if we had to pool at stage 2.

A necessary condition for (2.19) is that

$$(2.20) \qquad \frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1}$$

which reduces to the requirement that

$$(2.21) \qquad \frac{a_1 + T_{12}}{n_{12}} \geq \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} .$$

If (2.21) is not violated, we proceed to stage 3.

If (2.21) is violated, then we shall pool T_0 and T_{12} and n_0 and n_{12} to form

$$T_{0,12} \stackrel{\text{def}}{=} (T_0 + T_{12})/2 \text{ , and } n_{0,12} \stackrel{\text{def}}{=} (n_0 + n_{12})/2 \text{ ,}$$

and replace the appropriate quantities in (2.20) by their pooled values. Having done this, we shall need to have

$$(2.22) \qquad \frac{\alpha_0 + T_{0,12} + a_1 + T_{0,12}}{\beta_0 + n_{0,12} + n_{0,12} - 1} \geq \frac{\alpha_0 + T_{0,12}}{\beta_0 + n_{0,12} - 1}$$

which because of $a_1 \geq \alpha_0$ is always true.

Since $T_{0,12} = \frac{1}{2}\left(T_0 + \frac{1}{2}(T_1 + T_2)\right)$ and $n_{0,12} = \frac{1}{2}\left(n_0 + \frac{1}{2}(n_1 + n_2)\right)$, condition (2.22) reduces to

$$\frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{4\alpha_0 + 2T_0 + T_1 + T_2}{4\beta_0 + 2n_0 + n_1 + n_2 - 1} .$$

Thus, if we did not have to pool at stage 1, and if condition (2.21) is violated, our Bayes estimator of θ_0 conditioned on T_0, T_1, and T_2 is

$$E(\theta_0 \mid T_0, T_1, T_2) = \frac{4\alpha_0 + 2T_0 + T_1 + T_2}{4\beta_0 + 2n_0 + n_1 + n_2 - 1} . \tag{2.23}$$

The credibility intervals for θ_0 are now given by an inverted gamma distribution with scale parameter $(4\alpha_0 + 2T_0 + T_1 + T_2)/4$ and shape parameter $(4\beta_0 + 2n_0 + n_1 + n_2)/4$. We can now either stop or proceed to stage 3.

Reverting to Equation (2.20), we note that a necessary condition for satisfying this equation is that

$$\frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{2\alpha_0 + T_0 + T_1}{2\beta_0 + n_0 + n_1 - 1} . \tag{2.24}$$

If condition (2.24) is satisfied, we proceed to stage 3. Note that for $n_1 \geq n_2$ condition (2.24) reduces to

$$\frac{a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{\alpha_0 + T_1}{2\beta_0 + n_0 + n_1 - 1} ; \tag{2.25}$$

note that when $n_1 \geq n_2$, $2\beta_0 + n_0 + n_1 - 1 \geq \beta_0 + n_0 + n_{12} - 1$. Clearly, condition (2.25) is satisfied whenever $T_{12} \geq T_1$, that is, when $T_2 > T_1$.

Thus, in view of the above arguments, whenever condition (2.24) is violated, that is whenever $n_1 < n_2$ or $T_2 < T_1$, or both, we shall replace T_1 and n_1 by their pooled values T_{12} and n_{12} . Thus, after pooling, Equation (2.24) becomes

$$\frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{4\alpha_0 + 2T_0 + T_1 + T_2}{4\beta_0 + 2n_0 + n_1 + n_2 - 1}$$

which because of $a_1 \geq \alpha_0$ is always true.

To summarize, if we had to pool at stage 1, and if condition (2.24) is violated, our Bayes estimator of θ_0 conditioned on T_0 , T_1 , and T_2 is

$$E(\theta_0 \mid T_0, T_1, T_2) = \frac{4\alpha_0 + 2T_0 + T_1 + T_2}{4\beta_0 + 2n_0 + n_1 + n_2 - 1} .$$

Note that its estimator is identical to the one given by Equation (2.23) which was based on the fact that there was no pooling at stage 1.

However, in the present case, the credibility intervals for θ_0 are given by an inverted gamma distribution with scale parameters $(4\alpha_0 + 2T_0 + T_1 + T_2)/2$ and shape parameter $(4\beta_0 + 2n_0 + n_1 + n_2)/2$. We can now either stop or proceed to stage 3.

We contrast these, to the credibility intervals for θ_0 given after Equation (2.23), which pertained to the case of no pooling at stage 1. We note that the pooling at stage 1 and at stage 0 has a tendency to make the credibility intervals wider than those obtained when there is pooling at stage 0 only. *Thus, based on the above analysis, we claim that excessive pooling results in wider credibility intervals.*

Our analysis of the failure data at the succeeding stages, follows along the lines mentioned above.

2.3 Some Remarks on the Pooling Procedure

It is fairly clear that condition (2.7) is likely to be violated whenever (T_1/n_1) is not much larger than (T_0/n_0) . Note that (T_i/n_i) , $i = 0,1$, is the (non-Bayesian) maximum likelihood estimator of θ_i , $i = 0,1$. Thus (2.7) will be violated if the improvement in reliability in going from stage 0 to stage 1 is not significantly large. Thus,

pooling will be necessary whenever the effect of the design changes is not substantial (or if the design changes have produced a significant deterioration).

The pooling procedure advocated by us is one among several others that can be used. For example, we could have pooled the estimated mean times to failure (T_0/n_0) and (T_1/n_1), or we could have just pooled the observed total time on test T_0 and T_1. Irrespective of how we pool, the important question here is whether pooling the data is a legitimate Bayesian procedure.

A pure and orthodox Bayesian might argue that by pooling we have violated the "likelihood principle" of statistical inference. He will take objections on the grounds that our decision rule is not based on the information provided to us by the true posterior distribution, but instead, is based on a posterior distribution which is modified to suit our hypothesis. He would recommend that instead of pooling, it would be better to choose $a_1 >> \alpha_0$, so that condition (2.7) will always be satisfied, or to choose a joint prior distribution on θ_0 and θ_1 in such a manner that there is no probability mass in the region $\theta_1 < \theta_0$ (as has been done by Barlow et al. (1972)).

Our response to the above arguments is that not allowing any prior or posterior probability in the region $\theta_1 < \theta_0$ is too strong, and perhaps an unreasonable requirement, and that pooling is necessitated by the randomness of the data. Thus, whenever the posterior distributions violate our requirement, that is, $(\theta_1 \mid T_1) \overset{st}{\geq} (\theta_0 \mid T_0)$, it is preferable to pool the variables rather than to change the prior parameters in order to make $a_1 >> \alpha_0$. As a compromise, we may want to delete the requirement that $\theta_1 \overset{st}{\geq} \theta_0$ with respect to the posterior distributions, and just work with the requirement that $(\theta_1) \overset{st}{\geq} (\theta_0 \mid T_0)$ [see Section 4].

Another major comment about our procedure pertains to our rational for requiring that subsequent to pooling conditions of the type given by Equations (2.11) be satisfied. Note that (2.11) is analogous in appearance to the necessary condition (2.6.1). However, the terms which comprise condition (2.11) are not the means of the true posterior distribution after pooling. For instance, after we replace T_0 by T_{01}, and n_0 by n_{01}, the mean of the posterior distribution of θ_0 conditioned on T_{01} is *not* $(\alpha_0 + T_{01})/(\beta_0 + n_{01} - 1)$, as is implied by the right hand side of (2.11). The actual mean of the true posterior distribution is quite complicated, and is given by

$$\int_{\text{all } \theta_1} \theta_1 g(\theta_0, \theta_1 \mid T_{01}) d\theta_1$$

where $g(\theta_0, \theta_1 \mid T_{01})$ is the joint posterior distribution of θ_0 and θ_1, given by

$$\left(\frac{1}{\Gamma(n_{01} + \beta_0)\Gamma(n_{01} + \beta_1)} \frac{1}{\theta_0^{n_{01}+\beta_0+1} \theta_1^{n_{01}+\beta_1+1}} e^{-\left(\frac{\alpha_0}{\theta_0} + \frac{\alpha_1}{\theta_1}\right)}\right)$$

$$\frac{\displaystyle\int_0^{2T_{01}} (2T_{01} - v)^{n_{01}-1} v^{n_{01}-1} e^{-\left(\frac{2T_{01}-v}{\theta_0} + \frac{v}{\theta_1}\right)} dv}{\displaystyle\int_0^{2T_{01}} \frac{(2T_{01} - v)^{n_{01}-1} v^{n_{01}-1}}{(\alpha_0 + 2T_{01} - v)^{n_{01}+\beta_0} (\alpha_1 + v)^{n_{01}+\beta_1}} dv} .$$

In view of the computational difficulties involved with the above equations, we choose $(\alpha_0 + T_{01})/(\beta_0 + n_{01} - 1)$ as being *analogous* to the mean of the posterior distribution of θ_0 given T_{01}. We approximate the mean of the posterior distribution of θ_1 given T_{01} in a similar manner, and thus write condition (2.11). Since the above approximations have been motivated by the arguments which lead us to pool, we feel that they are inherently satisfactory.

We should close this section by stating that in the light of the above discussions, our approach should be called a "pseudo-Bayesian approach."

3. AN ISOTONIC REGRESSION OF THE RAW POSTERIOR MEANS

Our Strategy 1 of Section 2.1 specifies that the inequality (2.9), and other similar inequalities be ignored whenever the posterior means do not have the correct order. As a result of the above strategy, we will have at the end of testing over the $(\tau + 1)$ stages the $(\tau + 1)$ posterior means

$$E(\theta_0 \mid T_0), E(\theta_1 \mid T_0, T_1), \ldots, E(\theta_\tau \mid T_0, T_1, \ldots, T_\tau)$$

where

$$E(\theta_0 \mid T_0) = \frac{\alpha_0 + T_0}{\beta_0 + n_0 - 1} ,$$

and

$$E(\theta_i \mid T_0, T_1, \ldots, T_i) = \frac{\alpha_0 + T_0 + \sum_{j=1}^{i} (a_j + T_j)}{\beta_0 + n_0 + \sum_{j=0}^{i} n_j - 1} , \quad i = 1, 2, \ldots, \tau .$$

Under our postulate of reliability growth, we would need to have (as a necessary condition)

$$(3.1) \quad E(\theta_1 \mid T_0, \ldots, T_i) \leq E(\theta_{i+1} \mid T_0, \ldots, T_{i+1}) , \quad i = 0, 1, \ldots, \tau - 1 .$$

If condition (3.1) is satisfied, then our Bayes estimate of the reliability growth curve is given by these posterior means, and our Bayes estimator of the reliability at stage τ, is simply $E(\theta_\tau \mid T_0, T_1, \ldots, T_\tau)$. Note that because of the adaptive nature of our scheme, $E(\theta_\tau \mid T_0, \ldots, T_\tau)$, is based on the failure data over all the previous and the present stages

of testing, and our prior knowledge about the magnitude of the improvement over each stage.

If condition (3.1) is violated by any one or more of the indices i , $i = 0, 1, \ldots, \tau$, then, we shall, following Barlow et al. (1972) pool the adjacent violators to obtain the isotonic regression of $E(\theta_i \mid T_0, \ldots, T_i)$, $i = 0, \ldots, \tau$, say, $E^*(\theta_i \mid T_0, \ldots, T_i)$. We shall use the $E^*(\theta_i \mid T_0, \ldots, T_i)$, $i = 0, \ldots, \tau$ as our estimate of the reliability growth and $E^*(\theta_\tau \mid T_0, \ldots, T_\tau)$ as our estimate of the reliability at stage τ . Note that like $E(\theta_\tau \mid T_0, \ldots, T_\tau)$, $E^*(\theta_\tau \mid T_0, \ldots, T_\tau)$ is based on the failure data over all the previous stages, our prior knowledge about the magnitude of the improvements at each stage, and the postulate of reliability growth.

The remarks of Section 2.3 are also appropriate for the isotonic regression estimators $E^*(\theta_i \mid T_0, \ldots, T_i)$, since

a) by performing an isotonic regression of the true posterior means we have violated the likelihood principle, and

b) the estimators $E^*(\theta_i \mid T_0, \ldots, T_i)$ not being the true posterior means of the θ_i , $i = 0, \ldots, \tau$, they are not fully Bayesian.

4. ESTIMATION WHEN THE ORDERING IS WITH RESPECT TO THE PRIORS ONLY

In Section 2, we have considered the case when the mean lifetimes were stochastically ordered with respect to both the prior and the posterior distributions. In this section, we shall delete the requirement that the means be ordered with respect to the posterior distribution. When this is done, we will not have to pool the violators, nor will we have to perform an isotonic regression of the posterior means, should we choose not to pool.

We start off by choosing a prior distribution of θ_0 , $g(\theta_0;\alpha_0,\beta_0)$ as given by Equation (2.0). The posterior distribution of θ_0 conditioned on T_0 , $g(\theta_0 \mid T_0;\alpha_0,\beta_0,n_0)$ is given by Equation (2.1). Assuming a squared error loss, the Bayes estimator of θ_0 is $E(\theta_0 \mid T_0)$, and this is given by Equation (2.2); the credibility intervals for θ_0 are given by Equation (2.3).

We shall now choose a prior distribution of θ_1 $g(\theta_1;\alpha_1,\beta_1)$ in such a manner that $\theta_1 \overset{st}{\geq} (\theta_0 \mid T_0)$. Following the discussion of Section 2, we shall choose $\alpha_1 = \alpha_0 + T_0 + a_1$ and $\beta_1 = \beta_0 + n_0$, where a_1 has the same interpretation as in Section 2. If at stage 1 we look upon T_0 as a constant, and assume that $g(\theta_1;\alpha_1,\beta_1)$ is independent of $g(\theta_0 \mid T_0;\alpha_0,\beta_0,n_0)$, then under the assumption of a squared error loss function, the Bayes estimator of θ_1 , conditioned on T_1 is $E(\theta_1 \mid T_1)$, given by Equation (2.6). The posterior distribution of θ_1 given T_1 , $g(\theta_1 \mid T_1;\cdot)$ is given by Equation (2.5), and the credibility intervals for θ_1 follow in the usual manner. Note that the above statements are only true if T_0 is viewed as a constant at stage 1, the prior distribution at stage 1 is assumed to be independent of the posterior distribution at stage 0.

Once we obtain $E(\theta_1 \mid T_1)$ we do not care to compare it with $E(\theta_0 \mid T_0)$, since we have not imposed any requirements on our parameters with respect to posterior distributions.

We now proceed to stage 2 by choosing our prior distribution of θ_2 in such a manner that

$$\theta_2 \overset{st}{\geq} (\theta_1 \mid T_1) .$$

Following our discussion in Section 2, we shall take $g(\theta_2;\alpha_2,\beta_2)$ as our prior distribution of θ_2 with $\alpha_2 = \alpha_0 + T_0 + a_1 + T_1 + a_2$, and $\beta_2 = \beta_0 + n_0 + n_1$; here again we shall treat T_0 and T_1 as constants. As before, if $g(\theta_2;\alpha_2,\beta_2)$ is taken to be independent of $g(\theta_1 \mid T_1;\cdot)$, then the mean of the posterior distribution of θ_2 conditioned on T_2 is our Bayes estimator of θ_2. We continue in this manner going from one stage to the next, obtaining at each stage the Bayes estimator of θ_i, $i = 3, \ldots, \tau$.

5. SUMMARY AND CONCLUSIONS

In this paper we have considered an adaptive approach for estimating reliability growth based on prior information.

In Section 2 we have imposed a strong requirement on our approach, by requiring that the mean times to failure at the various stages be stochastically ordered with respect to *both* the prior and the posterior distributions. The latter requirement can be satisfied if we pool the violators; however, pooling results in a violation of the likelihood principle, and other computational difficulties. Even though the computational difficulties can be avoided by using some approximations (see Section 2.3), the pooling makes our procedure not fully Bayesian. Thus, what we present in Section 2 can best be described as a *Bayesian scheme* for estimating reliability growth. A formal investigation of the properties of our scheme, despite the fact that it is not fully Bayesian, is an open question which we hope to address in our future work. Our scheme however, does produce results which are reasonable and intuitively satisfying.

Our review of the literature in Bayesian statistics indicates that there is no discussion or *even mention* of the problem estimating parameters which are stochastically ordered. As mentioned before, our strategy of pooling the violators to obtain the stochastic order may be unacceptable to a Bayesian. We therefore hope that this paper can stimulate some basic research into the general problem area mentioned above.

In view of the difficulties mentioned above, we, in Section 4, weaken the specifications on our approach by deleting the requirement that the parameters be stochastically ordered with respect to the posterior distributions. This simplification obviates the need for pooling the violators,

and thus would make our procedure fully Bayesian, and therefore optimal in the usual sense of minimizing the square error loss function. However, the adaptive nature of our problem imposes certain computational difficulties. We circumvent these by treating the observed statistic T_i to be constant at stage $(i + 1)$, $i = 0,1,2, \ldots, \tau - 1$, and by assuming the prior distribution at stage j to be independent of the posterior distribution at stage $j - 1$, $j = 1,2, \ldots, \tau$. Then, within the context of the above assumptions, our procedure of Section 4 is fully Bayesian.

5.1 Future Work

There are several other aspects of the reliability growth problem that we plan to address in our subsequent work. These are:

i) An evaluation of the gain in information obtained by considering an adaptive scheme wherein previous data obtained at stages $0,1, \ldots, i - 1$, is used in the estimation at stage i , versus a nonadaptive scheme wherein only the data at stage i is used. It is conceivable that an adaptive procedure will be advantageous whenever the improvement in reliability form one stage to another is small, whereas if there is a drastic change in reliability at a particular stage, then the data from the previous stages will tend to diminish its true effect.

ii) A cost-benefit analysis of the reliability growth procedure. That is, we would like to evaluate the trade-off between the costs incurred in improving the reliability at stage i versus the actual improvement in reliability at stage $(i + 1)$ say $(\theta_{i+1} - \theta_i) \stackrel{\text{def}}{=} \nabla\theta_i$, $i = 0,1, \ldots, \tau$. It is conceivable

that the $\nabla\theta_i$ will be decreasing in i (there is only so much that one can do to improve a system) whereas the C_i will be either constant or increasing in i. What we need is a stopping rule which tells us when to stop performing the improvements on the system and put the system into operation, based on the costs C_i and out best estimate of $\nabla\theta_i$.

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